Facts about co sets

Suppose H is a subgroup of a group G.
let $a, b \in G$.

Question: What are the possibilities for
$aH \cap bH$?

Answer: either $aH \cap bH = \emptyset$ or $aH = bH = aH \cap bH$.

Proof: Suppose $aH \cap bH \neq \emptyset$, so that
$ah_1 = bh_2$ for some $h_1, h_2 \in H$.
$\Rightarrow a = bh_2h_1^{-1} \Rightarrow a \in bH$.
$\therefore aH \subseteq bH$.
Similarly $bH \subseteq aH$, so $aH = bH$.

Is it true if we just assume H is a mere subset of G?
No. Counterexample: $G = \mathbb{Z}_{10}$, let $H = \{1, 2, 3\}$

$a = 0$
$b = 1$

$a + H = \{0+1, 0+2, 0+3\}$
$b + H = \{1+1, 1+2, 1+3\} = \{2, 3, 4\}$
$(a+H) \cap (b+H) = \{2, 3\} \neq a + H$
$\neq b + H$.

Corollary of this fact $\Rightarrow$ If $H < G$, then
the cosets of H partition G.

$$G = \bigcup_{a \in G} aH = \bigcup_{a \in A} aH = \coprod_{a \in A} aH$$

disjoint union.

A = collection of elements of G s.t. each belongs to unique coset.

Item of note: if H is finite, then $\forall a \in G$,

$|aH| = |H|$ and so all cosets have the same # of elements.

i.e. there is a 1-1 correspondence $F: H \to aH$

given by $F(x) = ax$.

Observe it's certainly onto, and

if $x, y \in H$ and $F(x) = F(y)$, then

$$ax = ay \Rightarrow a^{-1}ax = a^{-1}ay$$
$$\Rightarrow x = y,$$

so F is 1-1.

$\therefore |aH| = |H|$.

[Lagrange's Theorem] Thus, if G is a finite group, say $o(G) = n$, then if $H < G$, then $o(H) \mid o(G)$.

Proof: Let A be a collection $\{a_j\}$ of elements of G s.t. each a_j is in a unique coset a_jH, and $\bigcup_j a_j H = G$.

then by the above, this a disjoint union, so

$$|G| = \sum_j |a_j H| = \sum_j |H| = m|H|,$$

where m is # of distinct cosets. $\blacksquare$

Above $m = |G/H| = [G:H] =$ "index of H in G".

Order of a group element:

For a in a group G,

$o(a) =$ least positive integer m s.t. $a^m = e$ = identity of G.

$o(a) \in \mathbb{N} \cup \{\infty\}$.

Corollary of Lagrange's Theorem: If G is a finite group and $a \in G$, then $o(a) \mid o(G)$.

Pf. let $H = \langle a \rangle = \{e, a, a^2, \ldots, a^{m-1}\}$ is a subgroup (cyclic subgroup generated by a) of G, and $m = o(a) = |H|$.

Lagrange's thm tells us that

$o(H) = m \mid o(G)$. ∎

If $H < G$, then G/H = set of left cosets of H

G/H is a group if H is a normal subgroup, using

$(aH)(bH) = aHbH = abHH = abH$

If H is not normal, $a, b \in G$, it is not necessarily true that $(aH)(bH) = cH$ for some c.

Aside: Group actions on sets.

Let G be a group, and let S be a set.

We say G acts on S if $\forall g \in G$,

there is a map $g\cdot : S \to S$

$$s \mapsto g \cdot s$$

such that $\forall g_1, g_2 \in G$,

$$g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s .$$

(Definition of Left action of G on S)

Other notations: $L_{g_1}(s) = g_1 \cdot s$

If we defined right action, $(R_{g_1} \circ R_{g_2})(s)$

$$= R_{g_1}(R_{g_2}(s)) = R_{g_1}(s \cdot g_2)$$

$$= (s \cdot g_2) \cdot g_1 = s \cdot (g_2 g_1)$$

$$= R_{g_2 g_1}(s).$$

One example: G acts on G/H by the following. If $g \in G$, $a \in G$, then

$$g(aH) = (ga)H.$$

Then $\forall g_1, g_2 \in G$, $g_1(g_2(aH)) = g_1((g_2 a)H)$

$$= g_1(g_2 a) H = (g_1 g_2) a H = (g_1 g_2)(aH).$$

$\therefore$ This is a left action.

Another example: let $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$.

Then $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on S^1 by

$$(a,b)\underbrace{(x,y)}_{\in S^1} = \underbrace{(\overset{\pm 1}{(1-2a)}x, \overset{\pm 1}{(1-2b)}y)}_{\in S^1}.$$

Where $\mathbb{Z}_2 = \{0,1\} \bmod 2$.

$a \in \mathbb{Z}_2 \quad 1-2a = \begin{cases} 1 & \text{if } a=0 \\ -1 & \text{if } a=1 \end{cases}$

Let's check if this is a group action: Suppose $a,b,c,d \in \mathbb{Z}_2$

$$(c,d)\left[(a,b)\underbrace{(x,y)}_{\in S^1}\right] = (c,d)\left((1-2a)x, (1-2b)y\right)$$

$$= \left((1-2c)(1-2a)x, (1-2d)(1-2b)y\right).$$

On the other hand,

$$\Big(\overset{\mathbb{Z}_2\times\mathbb{Z}_2}{(c,d)+(a,b)}\Big)(x,y) = \left((c+a, d+b)\right)(x,y)$$

$$= \left((1-2(c+a))\cdot x, (1-2(d+b))y\right)$$

$$= \left((1-2c)(1-2a)x, \quad \right). \checkmark$$

sim.

Possibilities

a	c	$1-2(c+a)$	$(1-2a)(1-2c)$
0	0	1	1 ✓
0	1	-1	-1 ✓
1	0	-1	-1 ✓
1	1	1	1 ✓